1. This follows directly by the addition formulae.
2. We have already seen a proof of this formula during the lectures.
3. Thanks to Fubini's theorem, we have

$$\begin{aligned}|a_n|^2 &= \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(\theta)f(\varphi)\cos(n\theta)\cos(n\varphi)d\theta\,d\varphi\\ |b_n|^2 &= \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(\theta)f(\varphi)\sin(n\theta)\sin(n\varphi)d\theta\,d\varphi.\end{aligned}$$

The first formula therefore follows from the addition formula

$$\cos(n\theta)\cos(n\varphi)+\sin(n\theta)\sin(n\varphi)=\cos(n(\theta-\varphi)).$$

Since

$$\int_0^{2\pi}\cos(n(\theta-\varphi))d\theta=\int_0^{2\pi}\cos(n(\theta-\varphi))d\varphi=0,$$

we deduce that

$$\int_0^{2\pi}\int_0^{2\pi}|f(\theta)|^2\cos(n(\theta-\varphi))d\theta\,d\varphi=\int_0^{2\pi}\int_0^{2\pi}|f(\varphi)|^2\cos(n(\theta-\varphi))d\theta\,d\varphi=0$$

and the second formula follows by Fubini's theorem.

4. By Parseval's identity and Fubini's theorem, we have

$$\begin{aligned}\frac{1}{2}\int_{B(0,r)}|\nabla u|^2dx &= \frac{1}{2}\int_0^r\int_0^{2\pi}\left(|\partial_\rho u|^2+\frac{1}{\rho^2}|\partial_\theta u|^2\right)\rho\,d\rho\,d\theta\\ &= \pi\sum_{n=1}^{\infty}\int_0^r n^2\left(|a_n|^2+|b_n|^2\right)\rho^{2n-1}d\rho=\frac{\pi}{2}\sum_{n=1}^{\infty}n\,r^{2n}\left(|a_n|^2+|b_n|^2\right).\end{aligned}$$

5. The identity simply follows from the identity in 3. and Fubini's theorem applied to the expression in 4.
6. We have

$$-2\,Q(r,\alpha)=\sum_{n=1}^{\infty}n\,r^{2n}e^{in\alpha}+\sum_{n=1}^{\infty}n\,r^{2n}\,e^{-in\alpha}.$$

Write to simply $z=r^2e^{i\alpha}$. We then find that

$$\begin{aligned}-2\,Q(r,\alpha) &= z\sum_{n=1}^{\infty}n\,z^{n-1}+\overline{z}\sum_{n=1}^{\infty}n\,\overline{z}^{n-1}=\frac{z}{(1-z)^2}+\frac{\overline{z}}{(1-\overline{z}^2)}\\ &= \frac{2\,\mathrm{Re}\,(z)(1+|z|)^2-4|z|^2}{\left(1+|z|^2-2\,\mathrm{Re}\,(z)\right)^2}.\end{aligned}$$

We deduce that

$$Q(r,\alpha)=r^2\frac{a-b}{(a+b)^2}\;\text{ where }\;a=(1+r^2)^2\sin^2\left(\frac{\alpha}{2}\right)\;\text{ and }\;b=(1-r^2)^2\cos^2\left(\frac{\alpha}{2}\right).$$

Finally, we get

$$\frac{Q(r,\alpha)}{Q(1-\alpha} = \frac{4r^2\sin^2\left(\frac{\alpha}{2}\right)}{a+b}\frac{a-b}{a+b} \text{ for all } \alpha \neq 0 \pmod{2\pi}.$$

We deduce that $Q(r,\alpha) \leq Q(1,\alpha)$ for all $0 \leq r < 1$ and that $Q(r,\alpha) \underset{r\to 1}{\longrightarrow} Q(1,\alpha)$ for all $\alpha \neq 0 \pmod{2\pi}$.

7. Using Lebesgue's dominated convergence, we deduce that

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} Q(r,\theta-\varphi)|f(\theta)-f(\varphi)|^2 d\theta\, d\varphi \underset{r\to 1}{\longrightarrow} D(f) < \infty,$$

and since by monotone convergence, we have $E_r(u) \underset{r\to 1}{\longrightarrow} E_1(u) = E(u)$, we deduce by the previous expression that $E(u) = D(f) < \infty$.

8. We estimate directly

$$\begin{aligned}\int_{R(\varepsilon)} Q(1,\theta-\varphi)|f(\theta)-f(\varphi)|^2 d\theta\, d\varphi &\leq \lim_{r\to 1}\int_{R(\varepsilon)} Q(r,\theta-\varphi)|f(\theta)-f(\varphi)|^2 d\theta\, d\varphi \\ &\leq \limsup_{\to 1}\int_0^{2\pi}\int_0^{2\pi} Q(r,\theta-\varphi)|f(\theta)-f(\varphi)|^2 d\theta\, d\varphi \\ &= \limsup_{r\to 1} 4\pi\, E_r(u) = 4\pi\, E(u) < \infty.\end{aligned}$$

Therefore, we deduce that

$$D(f) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{R(\varepsilon)} Q(1,\theta-\varphi)|f(\theta)-f(\varphi)|^2 d\theta\, d\varphi < \infty$$

and we can apply the previous proof to show that $E(u) = D(f)$, which concludes the proof of the theorem.